Stably weakly coreflective subcategories of the category of acts over a monoid

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Projective covers

An epimorphism $g : P \to A$ is called a **projective cover** of A if

- P is projective, and
- g restricted to any proper subact of P is not an epimorphism.

Theorem, Bass 1960

Given any ring R, the following are equivalent:

- R is (right) perfect (every right R-module has a projective cover).
- R satisfies DCC on principal (left) ideals.
- Every flat R-module is projective.

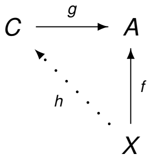
Theorem, Fountain 1976

Given any monoid S, the following are equivalent:

- S is (right) perfect (every right S-act has a projective cover).
- S satisfies DCC on principal (left) ideals and Condition (A).
- Every strongly flat S-act is projective.

Alternative definition of cover

An homomorphism $g : C \to A$ with $C \in \mathcal{X}$ is called an $\mathcal{X}$**-precover** of A if every homomorphism $f : X \to A$ with $X \in \mathcal{X}$ can be factored through g,



and we call it an $\mathcal{X}$**-cover** of A whenever $f = g \Rightarrow h$ is an isomorphism.

We say that $\mathcal{X} \subseteq \mathcal{C}$ is a stably weakly coreflective subcategory of $\mathcal{C}$ if every object in $\mathcal{C}$ has an $\mathcal{X}$-cover.

Theorem

Let $\mathcal{P}$ be the class of all projective modules/acts, then
$g : C \to A$ is a $\mathcal{P}$-cover of A if and only if $g : C \to A$ is a projective cover of A.

Hence $\mathcal{P} \subseteq$ Mod-R (resp. $\mathcal{P} \subseteq$ Act-S) is a stably weakly coreflective subcategory if and only if the ring R (resp. the monoid S) is perfect.

Flat cover conjecture

- In 1981 E. Enochs asked the question: Is the class of flat modules a stably weakly coreflective subcatgory of the category of all modules (i.e. does every module have a flat cover?)
- He showed that if a module has a flat precover, then it has a flat cover.
- He showed some classes of rings for which the conjecture was true.
- In 1995 J. Xu enlarged the class of rings for which the conjecture was known to be true to certain types of commutative Noetherian rings.
- The conjecture was finally proved for all rings independently by Encohs and Bican & El Bashir and published in a joint paper in 2001.

Flat covers of acts

free $\Rightarrow$ projecitve $\Rightarrow$ strongly flat $\Rightarrow$ Condition (P) $\Rightarrow$ flat $\Rightarrow \cdots$

Theorem (B & R, 2011)

Let S be a monoid, and $\mathcal{X}$ a class of S-acts closed under direct limits. If an S-act A has an $\mathcal{X}$-precover, then A has an $\mathcal{X}$-cover.

Let $\mathcal{SF}/\mathcal{CP}/\mathcal{F}$ be the class of strongly flat/Condition (P)/flat acts, since these are all closed under direct limits we have the following:

Corollary

If A has an $\mathcal{SF}/\mathcal{CP}/\mathcal{F}$-precover then it has an $\mathcal{SF}/\mathcal{CP}/\mathcal{F}$-cover.

Theorem (Bridge, 2010)

Let $\mathcal{X}$ be a class of S-acts such that $\coprod_{i \in I} X_i \in \mathcal{X} \Leftrightarrow X_i \in \mathcal{X}$ for each $i \in I$. Let S be a monoid that has only a set of indecomposable S-acts with property $\mathcal{X}$, then every S-act has an $\mathcal{X}$-precover.

Sketch proof

Let $\{X_i : i \in I\}$ be a set of indecomposable S-acts with property $\mathcal{X}$, and let each $(X_i)_f \cong X_i$. Then we have the following $\mathcal{X}$-precover.

$$\begin{array}{ccc} \coprod_{i \in I; f \in (B,A)} (X_i)_f & \longrightarrow & A \\ & \nwarrow & \uparrow f \\ & & \coprod_j X_j = B \end{array}$$

Results

Theorem (B & R, 2011)

Let S be a monoid that satisfies Condition (A), then every S-act has an $\mathcal{SF}/\mathcal{CP}$-cover.

Proof.

- Every strongly flat/Condition (P) act is a coproduct of locally cyclic acts
- Condition (A) $\Leftrightarrow$ locally cyclic acts are cyclic
- The cardinality of a cyclic act is bounded $|S/\rho| \leq |S|$
- The class of all indecomposable strongly flat/Condition (P) acts is a set
- $\mathcal{SF}/\mathcal{CP}$ is closed under direct limits.

□

Hence if every act has a strongly flat / Condition (P) cover [Khosravi et. al], then every act has an $\mathcal{SF}/\mathcal{CP}$-cover.

Results

Example (B & R, 2011)

There exist monoids that have a proper class (not a set) of indecomposable strongly flat acts, for example the full transformation monoid of an infinite set.

Open problems

Question

Does every S-act have an $\mathcal{SF}$-cover?

Question

What about other classes of acts, e.g. injective? (Enochs showed that every module has an injective cover if and only if the ring is Noetherian.)

Question

What about a dual theory for envelopes?