# Torsion free covering acts

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# Overview of results

Modules over Rings

Given a ring $R$, an $R$-module $M$ is called **torsion free** if $mr = 0$ implies $m = 0$ or $r = 0$ for all $m \in M$, $r \in R$.

**Theorem (Enochs, 1963)**

*Every module over an integral domain has a torsion free cover.*

Acts over Monoids

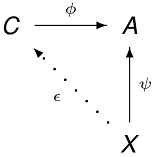
Given a monoid $S$, an $S$-act $A$ is called **torsion free** if $xc = yc$ implies $x = y$ for all $x, y \in A$, right cancellable $c \in S$.

**Theorem (B & R, 2011)**

*Every act over a cancellative monoid has a torsion free cover.*

## What is a cover?

An $\mathcal{X}$-cover has a two part definition as follows:
Let $\mathcal{C}$ be a category (e.g. **Mod-R**, **Act-S**) and $\mathcal{X}$ any full subcategory of $\mathcal{C}$ (e.g. objects that are projecitve, flat, torsion free, etc).
An homomorphism $\phi : C \to A$ with $C \in \mathcal{X}$ is called an $\mathcal{X}$**-precover** of $A$ if every homomorphism $\psi : X \to A$ with $X \in \mathcal{X}$ can be factored through $C$,




and we call it an $\mathcal{X}$**-cover** whenever $\psi = \phi$ forces $\epsilon$ to be an isomorphism. (Note, this is slightly weaker than the concept of a coreflective subcategory).

When $\mathcal{X} = \mathcal{P}$ the class of projective objects, this just becomes the usual definition of a projective cover. The dual definition with injective objects is the usual injective envelope.

## Summary of Enochs' proof

- He first proved that it was sufficient to show every torsion free and injective module factored through a torsion free module.
- He then made use of the fact that a torsion free, injective module over an integeral domain is isomorphic to a direct sum of copies of the field of fractions $K$, hence $\bigoplus_{f \in \mathrm{Hom}(K,M)} (K)_f$ is a torsion free precover of $M$

$$\begin{array}{ccc} \displaystyle\bigoplus_{f \in \mathrm{Hom}(K,M)} (K)_f & \longrightarrow & M \\ & \nwarrow & \uparrow \\ & & \displaystyle\bigoplus_{i \in I} (K)_i \cong T. \end{array}$$

- He finally proved that if $M$ has a torsion free precover, then it has a torsion free cover.

# Existence of $\mathcal{X}$-covers

**Theorem (B & R, 2011)**

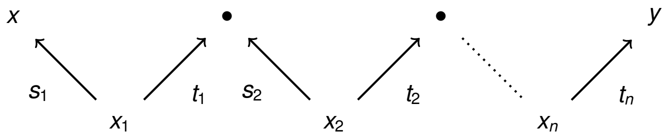
*Let $\mathcal{X}$ be a class of S-acts closed under direct limits. If A has an $\mathcal{X}$-precover, then A has an $\mathcal{X}$-cover.*

Torsion free acts are closed under direct limits, hence we need only show that every act has a torsion free precover.

# Decomposition of acts

An $S$-act $A$ is **decomposable** if it can be written as the coproduct of two subacts, $A = B \amalg C$, and **indecomposable** otherwise.

Given an indecomposable $S$-act $A$, for all $x, y \in A$, there exists $x_1, \ldots, x_n \in A$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S$ such that $x = x_1 s_1$, $x_1 t_1 = x_2 s_2$, $\ldots$, $x_n t_n = y$.




**Theorem**

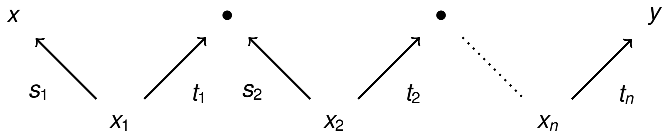
*Every $S$-act uniquely decomposes as a coproduct of indecomposable $S$-acts.*

# Existence of torsion free precovers

**Lemma (B & R, 2011)**

*Over a (right) cancellative monoid, there is only a set of indecomposable torsion free (right) acts.*

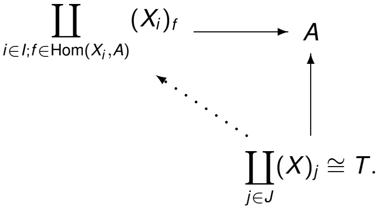
**Proof:** Let $X$ be an indecomposable torsion free $S$-act, then for all $x, y \in X$, we have the following diagram:




Clearly the number of arrows 'coming out' of each vertex is bounded by $|S|$. Since $S$ is right cancellative, $xs = x's$ implies $x = x'$ for all $x \in X$, $s \in S$. So the number of arrows 'coming in' to each vertex is also bounded by $|S|$. Therefore the cardinality of $X$ is bounded by $\max\{\aleph_0, |S|\}$ and so there is only a set of such $X$. $\square$

# Existence of torsion free precovers

Since there is only a set of torsion free indecomposable $S$-acts, $\{X_i : i \in I\}$, we have that $\coprod_{i\in I, f\in \mathrm{Hom}(X_i,A)}(X_i)_f$ is a torsion free precover of $A$




So we have proved,

**Theorem (B & R, 2011)**

*Over a (right) cancellative monoid, every (right) act has a torsion free cover.*

# Flat covers

A (right) $S$-act $A$ is called

- **flat** if $A \otimes -$ preserves all monomorphisms
- **weakly flat** if $A \otimes -$ preserves all embeddings of (left) ideals into $S$
- **principally weakly flat** if $A \otimes -$ preserves embeddings of principal (left) ideals into $S$

It is well known that

$$\text{flat} \Rightarrow \text{weakly flat} \Rightarrow \text{principally weakly flat} \Rightarrow \text{torsion free},$$

and in general these are strict implications.

**Theorem (Bulman-Fleming, McDowell 1990)**

*Over a right cancellative monoid $S$, an $S$-act is torsion free if and only if it is principally weakly flat.*

**Corollary (B & R, 2011)**

*Over a (right) cancellative monoid, every (right) act has a principally weakly flat cover.*